

Second Monthly Progress Report

on

Thermal Strain Analysis

of

Advanced Manned-Spacecraft Heat Shields

NASA Contract NAS 9-1986
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SECOND MONTHLY PROGRESS REPORT ON THERMAL STRAIN ANALYSIS
OF ADVANCED MANEUVERING-SPACECRAFT SHIELDS

NASA Contract NAS 9-1986

Phase A - Derivation of Basic Equations

This phase of the study, which constitutes the derivation of the basic equilibrium and stress equations in spherical and toroidal coordinates, is 95% completed. The only portion which is not included in this report (Appendix A) consists of defining certain coefficients in the equilibrium and stress equations which become singular on the axis of symmetry in the non-axisymmetric case. The singular point can be avoided in the numerical evaluation of the problem. However, if the number of grid nodes is limited by machine storage such that omission of the axis of symmetry would result in appreciable error in the remaining nodes, it is important to include this point. An analytical method has been developed for treating the singularity and it is anticipated that all of the necessary coefficients for the numerical evaluation will be obtained within one week.

Phase B - Finite Difference Formulation

All of the Phase B equations and partial differential equations have been derived for the grid system and are presented in this report (Appendix A). The remaining work for this phase consists of the finite difference formulation of the equations. The remaining work for this phase consists of the finite difference

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equations using the finite difference analogs. Since these coefficients will be generated with the aid of the IBM 7090, the work will require close coordination between the Project Scientist and Programmer and will be carried out as a joint effort.

Phase G - Report Preparation

This second monthly progress report constitutes 5% of the total report writing phase, which brings the total effort expended in this phase to 8%. In the next report (First Quarterly Progress Report), all of the work to date will be summarized in a complete, comprehensive report.

APPENDIX A (Continued)

Derivation of Equilibrium and Stress Equations
in Terms of Displacements in Spherical and Toroidal
Coordinates

Construction of Finite Difference Analogs to
Differential Equations

Summary

The work contained herein is a continuation of Appendix A presented in the First Monthly Progress Report, NASA Contract NAS 9-1986. Page numbers, figure numbers, table numbers and equation numbers are continued in consecutive order and reference is made to equations in the first progress report without repetition of the equations in this report. It should be noted that Fig. 1 refers to the two coordinate systems on the first page of Appendix A and Tables 1 and 2 refer to the coefficients of the equilibrium equations in spherical and toroidal coordinates, respectively, which were not labeled in the first report.

Equations for Stresses in Terms of Displacements

From Hooke's law, Eqs. (9) and (10),

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij} \left[\lambda \theta - (3\lambda + 2\mu) \int_{T_0}^T \alpha(T) dT \right] \quad (17)$$

where δ_{ij} is the Kronecker delta defined by

$$\begin{aligned} \delta_{ij} &= 1, & i=j \\ &= 0, & i \neq j \end{aligned}$$

and

$$\theta \equiv e_{11} + e_{22} + e_{33}$$

Writing the strains in terms of displacements from either Eqs. (14) or (15) and shortening the nomenclature by defining the stresses

$$\tau_1 = \tau_{rr} \text{ or } \tau_{RR}$$

$$\tau_2 = \tau_{\varphi\varphi}$$

$$\tau_3 = \tau_{\theta\theta}$$

$$\tau_4 = \tau_{r\varphi} \text{ or } \tau_{\varphi r}$$

$$\tau_5 = \tau_{\varphi\theta}$$

$$\tau_6 = \tau_{r\theta} \text{ or } \tau_{\theta r}$$

Eq. (17) may be written in terms of displacements according to

$$\begin{aligned} \tau_l + \Delta_l (3\lambda + 2\mu) \int_{T_0}^T \alpha(T) dT &= \alpha_l u_r + \beta_l u_\varphi + \gamma_l u_\theta + \delta_l u \\ &+ \bar{\alpha}_l v_r + \bar{\beta}_l v_\varphi + \bar{\gamma}_l v_\theta + \bar{\delta}_l v \\ &+ \bar{\alpha}_l w_r + \bar{\beta}_l w_\varphi + \bar{\gamma}_l w_\theta + \bar{\delta}_l w \end{aligned} \quad (18)$$

Where

$$\begin{aligned} \Delta_l &= 1 & \text{if } l = 1, 2, 3 \\ &= 0 & \text{if } l = 4, 5, 6 \end{aligned}$$

Equations at the Axis of Symmetry

Certain of the coefficients in the displacement equilibrium and stress equations become singular at the axis of symmetry ($\varphi = 0$). For the non-axially-symmetric case the axis of symmetry has no special physical significance and this point can be avoided. For the axially-symmetric case, however, the axis of symmetry is generally quite important and the singular coefficients may be evaluated by the use of L'Hôpital's rule. For example, the coefficient H_1 in the displacement equilibrium equations in spherical coordinates is $\mu \cot \varphi / R^2$ which becomes infinite as φ approaches zero. From Eq. (16), this term multiplies the displacement component $\frac{\partial u}{\partial \varphi}$. The conditions for axial symmetry are

$$w(R, \varphi, \theta) = \frac{\partial w}{\partial \theta} = 0, \quad (19)$$

from which it can be shown that

$$v = \frac{\partial u}{\partial \varphi} = \frac{\partial^2 v}{\partial \varphi^2} \downarrow^0 \text{ at } \varphi = 0. \quad (20)$$

Hence, since $\frac{\partial u}{\partial \varphi}$ approaches zero while H_1 approaches infinity, L'Hôpital's rule is applicable to the product

$$\frac{\mu \cot \varphi}{R^2} \frac{\partial u}{\partial \varphi}$$

as $\varphi \rightarrow 0$. Taking the limit, there is obtained

$$\begin{aligned} \lim_{\varphi \rightarrow 0} \frac{\mu \cot \varphi}{R^2} \frac{\partial u}{\partial \varphi} &= \frac{\mu}{R^2} \lim_{\varphi \rightarrow 0} \frac{\frac{\partial u}{\partial \varphi} \cos \varphi}{\sin \varphi} \\ &= \frac{\mu}{R^2} \lim_{\varphi \rightarrow 0} \frac{\frac{\partial^2 u}{\partial \varphi^2} \cos \varphi - \frac{\partial u}{\partial \varphi} \sin \varphi}{\cos \varphi} \\ &= \frac{\mu}{R^2} \cdot \frac{\partial^2 u}{\partial \varphi^2} \end{aligned}$$

Hence, for this case, the coefficient H_1 becomes zero and the coefficient B_1 which multiplies $\frac{\partial^2 u}{\partial \varphi^2}$ is increased by μ/R^2 . Applying this limiting process to all the singular terms the following sets of coefficients are obtained:

Table 4
Coefficients of Equilibrium Equations on Axis
of Symmetry ($\varphi=0$) for Axially-Symmetric Case
Spherical Coordinates

l	1	2 ⁰	3 ⁰	l	1	2 ⁰	3 ⁰	l	1 ⁰	2 ⁰	3 ⁰
A_l	$\lambda+2\mu$ ①	0	0	\bar{A}_l	0 ①	0	0	\bar{A}_l	0	0	0
B_l	$2\mu/R$ ②	0	0	\bar{B}_l	0 ②	0	0	\bar{B}_l	0	0	0
C_l	0 ③	0	0	\bar{C}_l	0 ③	0	0	\bar{C}_l	0	0	0
D_l	0 ④	0	0	\bar{D}_l	$2(\lambda+\mu)/R$ ④	0	0	\bar{D}_l	0	0	0
E_l	0 ⑤	0	0	\bar{E}_l	0 ⑤	0	0	\bar{E}_l	0	0	0
F_l	0 ⑥	0	0	\bar{F}_l	0 ⑥	0	0	\bar{F}_l	0	0	0
G_l	$2(\lambda+2\mu)/R$ ⑦	0	0	\bar{G}_l	0 ⑦	0	0	\bar{G}_l	0	0	0
H_l	0 ⑧	0	0	\bar{H}_l	$-2(\lambda+3\mu)/R^2$ ⑧	0	0	\bar{H}_l	0	0	0
I_l	0 ⑨	0	0	\bar{I}_l	0 ⑨	0	0	\bar{I}_l	0	0	0
J_l	$-2(\lambda+2\mu)/R^2$ ⑩	0	0	\bar{J}_l	0 ⑩	0	0	\bar{J}_l	0	0	0

Table 5
Coefficients of Stress Equations on Axis of
Symmetry ($\varphi=0$) for Axially-Symmetric Case
Spherical Coordinates

l	1	2	3	4	5	6
α_l	$\lambda+2\mu$ ①	λ ②	λ ③	0 ④	0 ⑤	0 ⑥
β_l	0 ⑦	0 ⑧	0 ⑨	μ/R ⑩	0 ⑪	0 ⑫
γ_l	0 ⑬	0 ⑭	0 ⑮	0 ⑯	0 ⑰	0 ⑱
δ_l	$2\lambda/R$ ⑲	$2(\lambda+\mu)/R$ ⑳	$2(\lambda+\mu)/R$ ㉑	0 ㉒	0 ㉓	0 ㉔
$\bar{\alpha}_l$	0 ㉕	0 ㉖	0 ㉗	μ ㉘	0 ㉙	0 ㉚
$\bar{\beta}_l$	$2\lambda/R$ ㉛	$2(\lambda+\mu)/R$ ㉜	$2(\lambda+\mu)/R$ ㉝	0 ㉞	0 ㉟	0 ㊱
$\bar{\gamma}_l$	0 ㊲	0 ㊳	0 ㊴	0 ㊵	0 ㊶	0 ㊷
$\bar{\delta}_l$	0 ㊸	0 ㊹	0 ㊺	$-\mu/R$ ㊻	0 ㊼	0 ㊽
$\bar{\alpha}_l$	0 ㊾	0 ㊿	0 ㋀	0 ㋁	0 ㋂	μ ㋃
$\bar{\beta}_l$	0 ㋄	0 ㋅	0 ㋆	0 ㋇	$0 \rightarrow \mu/R$ ㋈	0 ㋉
$\bar{\gamma}_l$	0 ㋊	0 ㋋	0 ㋌	0 ㋍	0 ㋎	0 ㋏
$\bar{\delta}_l$	0 ㋐	0 ㋑	0 ㋒	0 ㋓	0 ㋔	$-\mu/R$ ㋕

Temperature Dependence of Elastic Constants

If, in addition to the coefficient of thermal expansion, the elastic constants are strongly dependent on Temperature, then additional terms must be included in the displacement equilibrium equations to account for the spacial derivatives of these constants. Differentiating the stress component τ_{ii} with respect to coordinate x_i , for example, from Eq. (9), there is obtained

$$\begin{aligned} \frac{\partial \tau_{ii}}{\partial x_i} &= \lambda \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \lambda}{\partial x_i} + 2\mu \frac{\partial e_{ii}}{\partial x_i} + 2e_{ii} \frac{\partial \mu}{\partial x_i} \\ &\quad - (3\lambda + 2\mu) \alpha(T) \frac{\partial T}{\partial x_i} + \frac{\partial}{\partial x_i} (3\lambda + 2\mu) \int_{T_0}^T \alpha(\tau) d\tau \\ &= \theta \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial x_i} + 2e_{ii} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial x_i} + \frac{\partial}{\partial T} (3\lambda + 2\mu) \frac{\partial T}{\partial x_i} \int_{T_0}^T \alpha(\tau) d\tau \\ &\quad + \lambda \frac{\partial \theta}{\partial x_i} + 2\mu \frac{\partial e_{ii}}{\partial x_i} - (3\lambda + 2\mu) \alpha(T) \frac{\partial T}{\partial x_i}, \end{aligned} \quad (21)$$

where the first three terms to the right of the equal sign have not been accounted for in the coefficients of Eq. (16). Representing the additional terms by primed quantities, Eq. (16) becomes

$$\begin{aligned} (A_k + A'_k) \frac{\partial^2 u}{\partial x_k^2} + (B_k + B'_k) \frac{\partial^2 u}{\partial x_k^2} + \dots &= \frac{(3\lambda + 2\mu) \alpha(T)}{\sqrt{g_{kk}}} \frac{\partial T}{\partial x_k} \\ &\quad + \frac{1}{\sqrt{g_{kk}}} \frac{\partial}{\partial T} (3\lambda + 2\mu) \frac{\partial T}{\partial x_k} \int_{T_0}^T \alpha(\tau) d\tau, \quad k=1,2,3 \end{aligned} \quad (22)$$

The coefficients A'_k, B'_k, \dots are tabulated below for spherical and toroidal coordinates, and for the special point in spherical coordinates on the axis of symmetry for the case of axial symmetry.

* The form of the temperature distribution is determined by the boundary conditions
 from temperature coordinates

	$k=1$	$k=2$	$k=3$
A_k	0	0	0
B_k	0	0	0
C_k	0	0	0
D_k	0	0	0
E_k	0	0	0
F_k	0	0	0
G_k	$\frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial r}$	$\frac{1}{r} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{1}{a+r \sin \varphi} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$
H_k	$\frac{1}{r^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{1}{r} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial r}$	$\frac{1}{a+r \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial r}$
I_k	$\frac{1}{(a+r \sin \varphi)^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	0	$\frac{1}{a+r \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial r}$
J_k	$(\frac{1}{r} + \frac{\sin \varphi}{a+r \sin \varphi}) \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial r}$	$\frac{1}{r^2} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \varphi} + \frac{\sin \varphi}{r(a+r \sin \varphi)} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{\sin \varphi}{(a+r \sin \varphi)^2} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \varphi} + \frac{\sin \varphi}{r(a+r \sin \varphi)} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$
\bar{A}_k	0	0	0
\bar{B}_k	0	0	0
\bar{C}_k	0	0	0
\bar{D}_k	0	0	0
\bar{E}_k	0	0	0
\bar{F}_k	0	0	0
\bar{G}_k	$\frac{1}{r} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{\partial \mu}{\partial T} \frac{\partial T}{\partial r}$	$\frac{1}{r(a+r \sin \varphi)} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$
\bar{H}_k	$\frac{1}{r} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial r}$	$\frac{1}{r^2} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \varphi}$	$\frac{1}{r(a+r \sin \varphi)} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$
\bar{I}_k	0	$\frac{1}{(a+r \sin \varphi)^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{\cos \varphi}{(a+r \sin \varphi)^2} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \varphi}$
\bar{J}_k	$\frac{\cos \varphi}{a+r \sin \varphi} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial r} - \frac{1}{r^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{\cos \varphi}{r(a+r \sin \varphi)} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi} - \frac{1}{r} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial r}$	0
\bar{A}_k	0	0	0
\bar{B}_k	0	0	0
\bar{C}_k	0	0	0
\bar{D}_k	0	0	0
\bar{E}_k	0	0	0
\bar{F}_k	0	0	0
\bar{G}_k	$\frac{1}{a+r \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{1}{r(a+r \sin \varphi)} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{1}{r^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$
\bar{H}_k	0	$\frac{1}{r(a+r \sin \varphi)} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$	$\frac{1}{(a+r \sin \varphi)^2} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \varphi}$
\bar{I}_k	$\frac{1}{a+r \sin \varphi} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial r}$	$\frac{1}{r(a+r \sin \varphi)} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$	$-\frac{\partial \mu}{\partial T} \left(\frac{\partial T}{\partial r} \sin \varphi + \frac{\partial T}{\partial \varphi} \cos \varphi \right)$
\bar{J}_k	$-\frac{\sin \varphi}{(a+r \sin \varphi)^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	$-\frac{\cos \varphi}{(a+r \sin \varphi)^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	0

Table 6
Additional Terms in Coefficients of Equilibrium Equations
from Temperature Dependence of Elastic Constants
Spherical Coordinates

	$k=1$		$k=2$		$k=3$	
A_k	0	⊙	0	⊙	0	⊙
B_k	0	⊙	0	⊙	0	⊙
C_k	0	⊙	0	⊙	0	⊙
D_k	0	⊙	0	⊙	0	⊙
E_k	0	⊙	0	⊙	0	⊙
F_k	0	⊙	0	⊙	0	⊙
G_k	$\frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial R}$	⊙	$\frac{1}{R} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$	⊙	$\frac{1}{R \sin \varphi} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \theta}$	⊙
H_k	$\frac{1}{R^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	⊙	$\frac{1}{R} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial R}$	⊙	0	⊙
I_k	$\frac{1}{R^2 \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \theta}$	⊙	0	⊙	$\frac{1}{R \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial R}$	⊙
J_k	$\frac{2}{R} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R}$	⊙	$\frac{2}{R^2} \frac{\partial}{\partial T}(\lambda+\mu) \frac{\partial T}{\partial \varphi}$	⊙	$\frac{2}{R^2 \sin \varphi} \frac{\partial}{\partial T}(\lambda+\mu) \frac{\partial T}{\partial \theta}$	⊙
\bar{A}_k	0	⊙	0	⊙	0	⊙
\bar{B}_k	0	⊙	0	⊙	0	⊙
\bar{C}_k	0	⊙	0	⊙	0	⊙
\bar{D}_k	0	⊙	0	⊙	0	⊙
\bar{E}_k	0	⊙	0	⊙	0	⊙
\bar{F}_k	0	⊙	0	⊙	0	⊙
\bar{G}_k	$\frac{1}{R} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	⊙	$\frac{\partial \mu}{\partial T} \frac{\partial T}{\partial R}$	⊙	0	⊙
\bar{H}_k	$\frac{1}{R} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R}$	⊙	$\frac{1}{R^2} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \varphi}$	⊙	$\frac{1}{R^2 \sin \varphi} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \theta}$	⊙
\bar{I}_k	0	⊙	$\frac{1}{R^2 \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \theta}$	⊙	$\frac{1}{R^2 \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial R}$	⊙
\bar{J}_k	$\frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R} \frac{\cos \varphi}{R} - \frac{1}{R^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	⊙	$\frac{\cos \varphi}{R^2} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi} - \frac{1}{R} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial R}$	⊙	$\frac{\cos \varphi}{R^2 \sin \varphi} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \theta}$	⊙
\bar{A}_k	0	⊙	0	⊙	0	⊙
\bar{B}_k	0	⊙	0	⊙	0	⊙
\bar{C}_k	0	⊙	0	⊙	0	⊙
\bar{D}_k	0	⊙	0	⊙	0	⊙
\bar{E}_k	0	⊙	0	⊙	0	⊙
\bar{F}_k	0	⊙	0	⊙	0	⊙
\bar{G}_k	$\frac{1}{R \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \theta}$	⊙	0	⊙	$\frac{\partial \mu}{\partial T} \frac{\partial T}{\partial R}$	⊙
\bar{H}_k	0	⊙	$\frac{1}{R \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \theta}$	⊙	$\frac{1}{R^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi}$	⊙
\bar{I}_k	$\frac{1}{R \sin \varphi} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R}$	⊙	$\frac{1}{R^2 \sin \varphi} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial \varphi}$	⊙	$\frac{1}{R^2 \sin \varphi} \frac{\partial}{\partial T}(\lambda+2\mu) \frac{\partial T}{\partial \theta}$	⊙
\bar{J}_k	$-\frac{1}{R^2 \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \theta}$	⊙	$-\frac{\cos \varphi}{R \sin \varphi} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial R}$	⊙	$-\frac{1}{R} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \varphi} - \frac{\cos \varphi}{R^2} \frac{\partial \mu}{\partial T} \frac{\partial T}{\partial \theta}$	⊙

Axis of Symmetry with Axial Symmetry

The only non-zero terms in the coefficients of Table 6 on the axis of symmetry in the axially-symmetric case are the following :

$$G'_1 = \frac{\partial}{\partial T} (\lambda + 2\mu) \frac{\partial T}{\partial R}$$

$$J'_1 = \frac{2}{R} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R}$$

$$\bar{H}'_1 = \frac{2}{R} \frac{\partial \lambda}{\partial T} \frac{\partial T}{\partial R}$$

The integral term in Eq. (22) is also non-zero for the equilibrium equation corresponding to $k=1$.

Finite Difference Formulation

The difference analogs to the partial differential equations are constructed on a grid network as shown in Fig. 2, for which $\alpha_1 = \text{constant}$ lines are ordered by the subscript i , $\alpha_2 = \text{constant}$ lines by the subscript j , $\alpha_3 = \text{constant}$ lines by the subscript k , and the intersection of grid lines (nodes) by the triple subscript i, j, k .

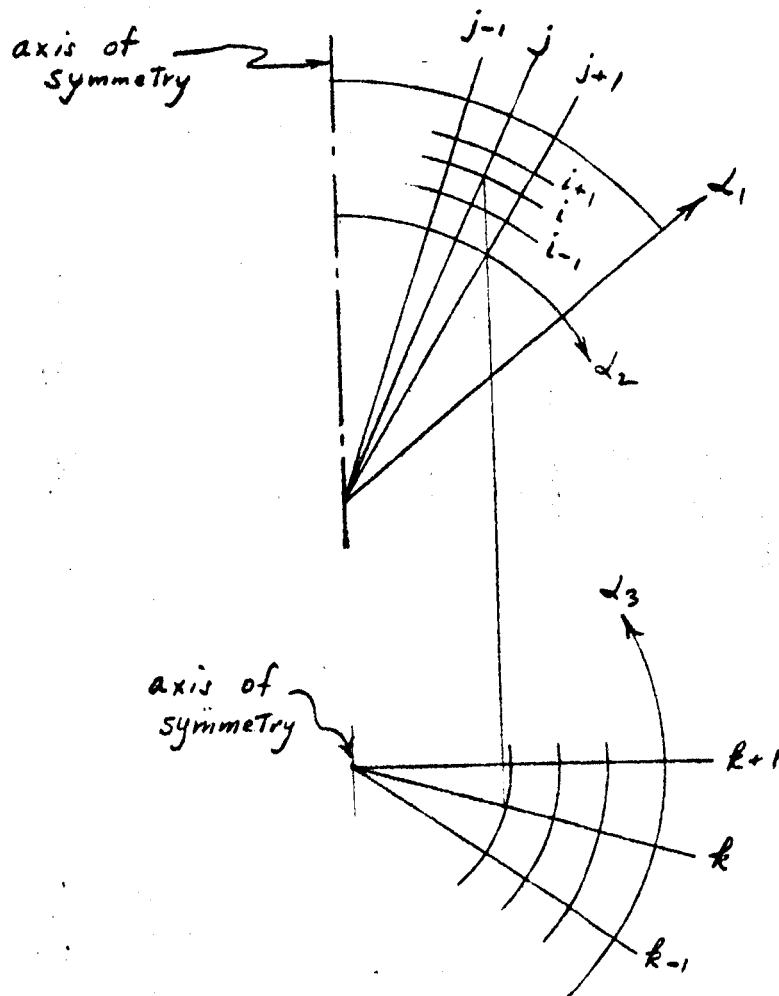


Fig 2 Grid Notation for
Finite Difference Formulation

For the general case the grid spacing will be irregular and the increments in the vicinity of a node will be designated by the following:

$$\begin{aligned} h_{11} &= (x_1)_{i+1} - (x_1)_i & h_{21} &= (x_2)_{i+1} - (x_2)_i & h_{31} &= (x_3)_{i+1} - (x_3)_i \\ h_{12} &= (x_1)_{i+2} - (x_1)_i & h_{22} &= (x_2)_{i+2} - (x_2)_i & h_{32} &= (x_3)_{i+2} - (x_3)_i \\ h_{13} &= (x_1)_i - (x_1)_{i-1} & h_{23} &= (x_2)_i - (x_2)_{i-1} & h_{33} &= (x_3)_i - (x_3)_{i-1} \\ h_{14} &= (x_1)_i - (x_1)_{i-2} & h_{24} &= (x_2)_i - (x_2)_{i-2} & h_{34} &= (x_3)_i - (x_3)_{i-2} \end{aligned}$$

Let $f(x_1, x_2, x_3)$ be any function of the coordinates such that it and its partial derivatives (up to any order required in the analysis) are continuous, and expand the function about the point i, j, k . Using a new coordinate system with origin at i, j, k and with ξ_1, ξ_2, ξ_3 directed along x_1, x_2, x_3 , respectively, the function $f(\xi_1, \xi_2, \xi_3)$ is written

$$\begin{aligned} f(\xi_1, \xi_2, \xi_3) &= f_{i,j,k} + B_1 \xi_1 + B_2 \xi_2 + B_3 \xi_3 + B_4 \xi_1 \xi_2 \\ &\quad + B_5 \xi_2 \xi_3 + B_6 \xi_3 \xi_1 + B_7 \xi_1^2 + B_8 \xi_2^2 + B_9 \xi_3^2 + \dots \\ &\quad + B_{10} \xi_1 \xi_2 \xi_3 + B_{11} \xi_1 \xi_2^2 + B_{12} \xi_1 \xi_3^2 + B_{13} \xi_1^2 \xi_2 \\ &\quad + B_{14} \xi_2 \xi_3^2 + \dots \end{aligned} \quad (23)$$

The first and second derivatives of $f(x_1, x_2, x_3)$ with respect to x_1, x_2, x_3 are obtained from Eq. (23) according to

$$\begin{aligned} \frac{\partial f}{\partial x_1} \Big|_{i,j,k} &= \frac{\partial f}{\partial \xi_1} \Big|_{(0,0,0)} = B_1 & \frac{\partial^2 f}{\partial x_1 \partial x_2} \Big|_{i,j,k} &= B_4 & \frac{\partial^2 f}{\partial x_1^2} \Big|_{i,j,k} &= 2B_7 \\ \frac{\partial f}{\partial x_2} \Big|_{i,j,k} &= \frac{\partial f}{\partial \xi_2} \Big|_{(0,0,0)} = B_2 & \frac{\partial^2 f}{\partial x_2 \partial x_3} \Big|_{i,j,k} &= B_5 & \frac{\partial^2 f}{\partial x_2^2} \Big|_{i,j,k} &= 2B_8 \\ \frac{\partial f}{\partial x_3} \Big|_{i,j,k} &= \frac{\partial f}{\partial \xi_3} \Big|_{(0,0,0)} = B_3 & \frac{\partial^2 f}{\partial x_3 \partial x_1} \Big|_{i,j,k} &= B_6 & \frac{\partial^2 f}{\partial x_3^2} \Big|_{i,j,k} &= 2B_9 \end{aligned} \quad (24)$$

(15)

By considering the values of $f(\xi_1, \xi_2, \xi_3)$ at the twelve nodes adjacent to i, j, k , the constants B_i are evaluated in terms of the function at these nodes and the grid spacings as shown in Fig. 3.

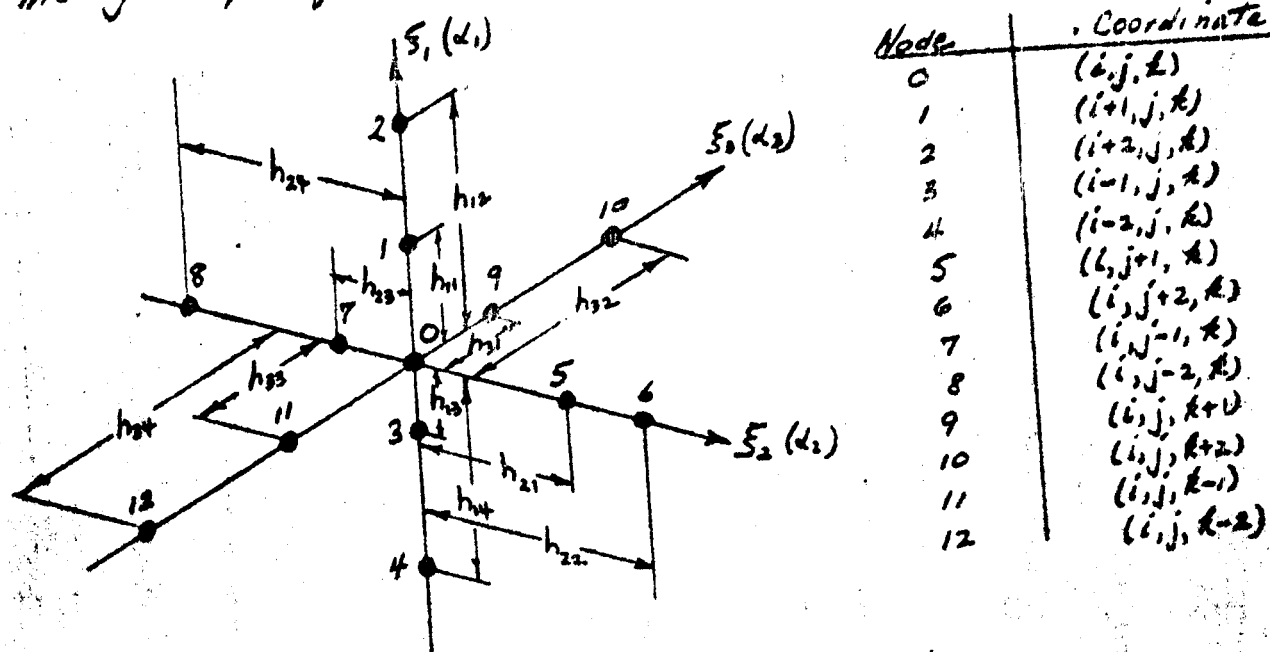


Fig. 3 Coordinates of Irregular Mesh Intervals

Note that the grid spacing increments h_{ij} do not, in general, have the dimensions of length but have the dimensions of d_1, d_2 and d_3 .

At points 1 and 3, Eq. (23) becomes

$$\left. \begin{aligned} f_{i+1,j,k} = f(h_{11}, 0, 0) &= f_{i,j,k} + B_1 h_{11} + B_7 h_{11}^2 \\ f_{i-1,j,k} = f(-h_{13}, 0, 0) &= f_{i,j,k} - B_1 h_{13} + B_7 h_{13}^2 \end{aligned} \right\} \quad (25)$$

where terms of higher order are deleted. Solving for B_1 and B_7 from Eqs. (25) gives for the first and second irregular central derivative with respect to d_1

$$\left. \begin{aligned} \frac{\partial f}{\partial d_1} \Big|_{i,j,k} &= \frac{h_{13}^2 f_{i+1,j,k} + (h_{11}^2 - h_{13}^2) f_{i,j,k} - h_{11}^2 f_{i-1,j,k}}{h_{11} h_{13} (h_{11} + h_{13})} \\ \frac{\partial^2 f}{\partial d_1^2} \Big|_{i,j,k} &= 2 \left[\frac{h_{13} f_{i+1,j,k} - (h_{11} + h_{13}) f_{i,j,k} + h_{11} f_{i-1,j,k}}{h_{11} h_{13} (h_{11} + h_{13})} \right] \end{aligned} \right\} \quad (26)$$

Substituting $h_{11} = h_{13} = h_1$ into Eqs. (26) gives for the first and second regular central derivatives with respect to x_1

$$\left. \begin{aligned} \frac{\partial f}{\partial x_1} \Big|_{i,j,k} &= \frac{f_{i+1,j,k} - f_{i-1,j,k}}{2h_1} \\ \frac{\partial^2 f}{\partial x_1^2} \Big|_{i,j,k} &= \frac{f_{i+1,j,k} - 2f_{i,j,k} + f_{i-1,j,k}}{h_1^2} \end{aligned} \right\} \quad (27)$$

By a similar procedure the following first and second regular and irregular central derivatives are obtained with respect to the coordinates x_2 and x_3 :

First Regular Central Derivatives ($h_2 = h_{21} = h_{23}$, $h_3 = h_{31} = h_{33}$)

$$\frac{\partial f}{\partial x_2} \Big|_{i,j,k} = \frac{f_{i,j+1,k} - f_{i,j-1,k}}{2h_2} \quad (28)$$

$$\frac{\partial f}{\partial x_3} \Big|_{i,j,k} = \frac{f_{i,j,k+1} - f_{i,j,k-1}}{2h_3} \quad (29)$$

First Irregular Central Derivatives

$$\frac{\partial f}{\partial x_2} \Big|_{i,j,k} = \frac{h_{23}^2 f_{i,j+1,k} + (h_{21}^2 - h_{23}^2) f_{i,j,k} - h_{21}^2 f_{i,j-1,k}}{h_{21} h_{23} (h_{21} + h_{23})} \quad (30)$$

$$\frac{\partial f}{\partial x_3} \Big|_{i,j,k} = \frac{h_{33}^2 f_{i,j,k+1} + (h_{31}^2 - h_{33}^2) f_{i,j,k} - h_{31}^2 f_{i,j,k-1}}{h_{31} h_{33} (h_{31} + h_{33})} \quad (31)$$

Second Regular Central Derivatives ($h_2 = h_{21} = h_{23}$, $h_3 = h_{31} = h_{33}$)

$$\frac{\partial^2 f}{\partial x_2^2} \Big|_{i,j,k} = \frac{f_{i,j+1,k} - 2f_{i,j,k} + f_{i,j-1,k}}{h_2^2} \quad (32)$$

$$\frac{\partial^2 f}{\partial x_3^2} \Big|_{i,j,k} = \frac{f_{i,j,k+1} - 2f_{i,j,k} + f_{i,j,k-1}}{h_3^2} \quad (33)$$

Second Irregular Central Derivatives

$$\left. \frac{\partial^2 f}{\partial x_1^2} \right)_{i,j,k} = \frac{2 [h_{23} f_{i,j+1,k} - (h_{21} + h_{23}) f_{i,j,k} + h_{21} f_{i,j-1,k}]}{h_{21} h_{23} (h_{21} + h_{23})} \quad (34)$$

$$\left. \frac{\partial^2 f}{\partial x_3^2} \right)_{i,j,k} = \frac{2 [h_{33} f_{i,j,k+1} - (h_{31} + h_{33}) f_{i,j,k} + h_{31} f_{i,j,k-1}]}{h_{31} h_{33} (h_{31} + h_{33})} \quad (35)$$

Forward and Backward Derivatives

By applying the same procedure as above with respect to two nodes located either forward or backward from the origin (i,j,k) , the first and second regular and irregular derivatives are obtained in terms of the function $f(x_1, x_2, x_3)$ evaluated at these nodes. The results are summarized below for the three coordinate directions:

First Irregular Forward Derivatives

$$\left. \frac{\partial f}{\partial x_1} \right)_{i,j,k} = \frac{-(h_{12}^2 - h_{11}^2) f_{i,j,k} + h_{12}^2 f_{i+1,j,k} - h_{11}^2 f_{i-1,j,k}}{h_{11} h_{12} (h_{12} - h_{11})} \quad (36)$$

$$\left. \frac{\partial f}{\partial x_2} \right)_{i,j,k} = \frac{-(h_{22}^2 - h_{21}^2) f_{i,j,k} + h_{22}^2 f_{i,j+1,k} - h_{21}^2 f_{i,j-1,k}}{h_{21} h_{22} (h_{22} - h_{21})} \quad (37)$$

$$\left. \frac{\partial f}{\partial x_3} \right)_{i,j,k} = \frac{-(h_{32}^2 - h_{31}^2) f_{i,j,k} + h_{32}^2 f_{i,j,k+1} - h_{31}^2 f_{i,j,k-1}}{h_{31} h_{32} (h_{32} - h_{31})} \quad (38)$$

First Regular Forward Derivatives

For equal grid spacings in each of the three coordinate directions, defined according to

$$\left. \begin{aligned} h_{11} &= h_{12}/2 \equiv h_1 \\ h_{21} &= h_{22}/2 \equiv h_2 \\ h_{31} &= h_{32}/2 \equiv h_3 \end{aligned} \right\} \quad (39)$$

Eqs. (34) - (38) reduce to

$$\left(\frac{\partial f}{\partial x_1}\right)_{i,j,k} = \frac{-3f_{i,j,k} + f_{i+1,j,k} - f_{i+2,j,k}}{2h_1} \quad (40)$$

$$\left(\frac{\partial f}{\partial x_2}\right)_{i,j,k} = \frac{-3f_{i,j,k} + 4f_{i,j+1,k} - f_{i,j+2,k}}{2h_2} \quad (41)$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{i,j,k} = \frac{-3f_{i,j,k} + 4f_{i,j,k+1} - f_{i,j,k+2}}{2h_3} \quad (42)$$

Second Irregular Forward Derivatives

$$\left(\frac{\partial^2 f}{\partial x_1^2}\right)_{i,j,k} = 2 \left[\frac{-h_{12}f_{i+1,j,k} + (h_{12}-h_{11})f_{i,j,k} + h_{11}f_{i+2,j,k}}{h_{11}h_{12}(h_{12}-h_{11})} \right] \quad (43)$$

$$\left(\frac{\partial^2 f}{\partial x_2^2}\right)_{i,j,k} = 2 \left[\frac{-h_{22}f_{i,j+1,k} + (h_{22}-h_{21})f_{i,j,k} + h_{21}f_{i,j+2,k}}{h_{21}h_{22}(h_{22}-h_{21})} \right] \quad (44)$$

$$\left(\frac{\partial^2 f}{\partial x_3^2}\right)_{i,j,k} = 2 \left[\frac{-h_{32}f_{i,j,k+1} + (h_{32}-h_{31})f_{i,j,k} + h_{31}f_{i,j,k+2}}{h_{31}h_{32}(h_{32}-h_{31})} \right] \quad (45)$$

Second Regular Forward Derivatives

With equal grid spacings, according to Eq. (39), Eqs. (43) - (45) reduce to

$$\left(\frac{\partial^2 f}{\partial x_1^2}\right)_{i,j,k} = \frac{-2f_{i+1,j,k} + f_{i,j,k} + f_{i+2,j,k}}{h_1^2} \quad (46)$$

$$\left(\frac{\partial^2 f}{\partial x_2^2}\right)_{i,j,k} = \frac{-2f_{i,j+1,k} + f_{i,j,k} + f_{i,j+2,k}}{h_2^2} \quad (47)$$

$$\left(\frac{\partial^2 f}{\partial x_3^2}\right)_{i,j,k} = \frac{-2f_{i,j,k+1} + f_{i,j,k} + f_{i,j,k+2}}{h_3^2} \quad (48)$$

First Irregular Backward Derivatives

$$\left(\frac{\partial f}{\partial x_1}\right)_{i,j,k} = \frac{h_{13}^2 f_{i-2,j,k} + (h_{14}^2 - h_{13}^2) f_{i,j,k} - h_{14}^2 f_{i-1,j,k}}{h_{13} h_{14} (h_{14} - h_{13})} \quad (49)$$

$$\left(\frac{\partial f}{\partial x_2}\right)_{i,j,k} = \frac{h_{23}^2 f_{i,j-2,k} + (h_{24}^2 - h_{23}^2) f_{i,j,k} - h_{24}^2 f_{i,j-1,k}}{h_{23} h_{24} (h_{24} - h_{23})} \quad (50)$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{i,j,k} = \frac{h_{33}^2 f_{i,j,k-2} + (h_{34}^2 - h_{33}^2) f_{i,j,k} - h_{34}^2 f_{i,j,k-1}}{h_{33} h_{34} (h_{34} - h_{33})} \quad (51)$$

First Regular Backward Derivatives ($h_{13} = \frac{h_{14}}{2} = h_1$, etc.)

$$\left(\frac{\partial f}{\partial x_1}\right)_{i,j,k} = \frac{f_{i-2,j,k} + 3f_{i,j,k} - 4f_{i-1,j,k}}{2h_1} \quad (52)$$

$$\left(\frac{\partial f}{\partial x_2}\right)_{i,j,k} = \frac{f_{i,j-2,k} + 3f_{i,j,k} - 4f_{i,j-1,k}}{2h_2} \quad (53)$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{i,j,k} = \frac{f_{i,j,k-2} + 3f_{i,j,k} - 4f_{i,j,k-1}}{2h_3} \quad (54)$$

Second Irregular Backward Derivatives

$$\left(\frac{\partial^2 f}{\partial x_1^2}\right)_{i,j,k} = 2 \left[\frac{h_{13} f_{i-2,j,k} + (h_{14} - h_{13}) f_{i,j,k} - h_{14} f_{i-1,j,k}}{h_{13} h_{14} (h_{14} - h_{13})} \right] \quad (55)$$

$$\left(\frac{\partial^2 f}{\partial x_2^2}\right)_{i,j,k} = 2 \left[\frac{h_{23} f_{i,j-2,k} + (h_{24} - h_{23}) f_{i,j,k} - h_{24} f_{i,j-1,k}}{h_{23} h_{24} (h_{24} - h_{23})} \right] \quad (56)$$

$$\left(\frac{\partial^2 f}{\partial x_3^2}\right)_{i,j,k} = 2 \left[\frac{h_{33} f_{i,j,k-2} + (h_{34} - h_{33}) f_{i,j,k} - h_{34} f_{i,j,k-1}}{h_{33} h_{34} (h_{34} - h_{33})} \right] \quad (57)$$

Second Regular Backward Derivatives

$$\left. \frac{\partial^2 f}{\partial x_1^2} \right|_{i,j,k} = \frac{f_{i-2,j,k} + f_{i,j,k} - 2f_{i-1,j,k}}{h_1^2} \quad (58)$$

$$\left. \frac{\partial^2 f}{\partial x_2^2} \right|_{i,j,k} = \frac{f_{i,j,k-2} + f_{i,j,k} - 2f_{i,j,k-1}}{h_2^2} \quad (59)$$

$$\left. \frac{\partial^2 f}{\partial x_3^2} \right|_{i,j,k} = \frac{f_{i,j,k-2} + f_{i,j,k} - 2f_{i,j,k-1}}{h_3^2} \quad (60)$$

Mixed Derivatives

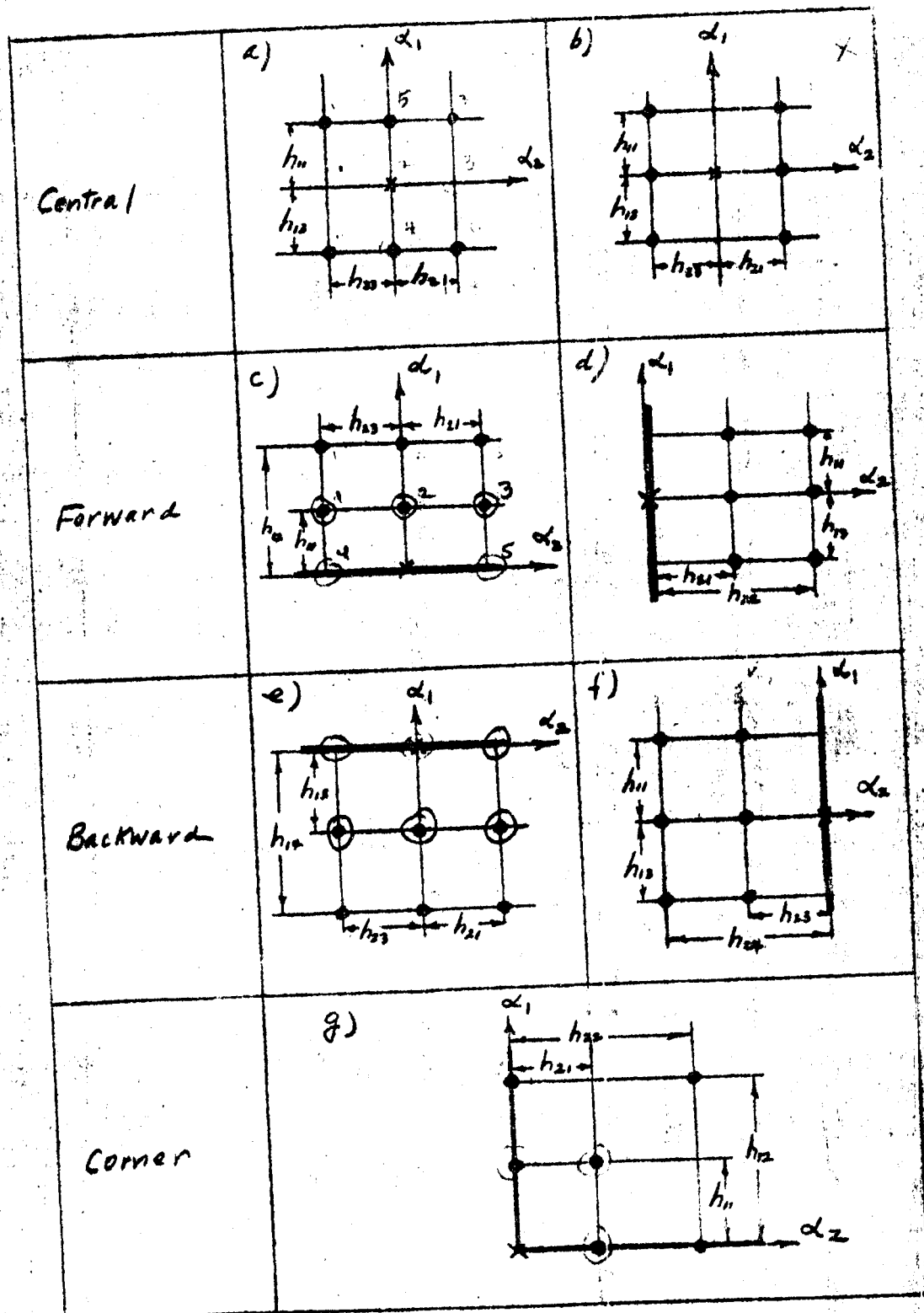
It can be shown from Eq. (23) that mixed derivatives require values of the function at any six nodes in the vicinity of the point under consideration. Fig. 3 shows various combinations of mixed derivatives with respect to the coordinate axes x_1 and x_2 . It is noted that the mixed central derivatives involve the four corner nodes as well as two adjacent nodes in either of the two coordinate directions. The various combinations shown in Fig. 3 are summarized below for the coordinate directions x_1 and x_2 :

Second Mixed Irregular Central Derivative with Respect to x_1 and x_2

$$a) \left. \frac{\partial^2 f}{\partial x_1 \partial x_2} \right|_{i,j,k} = \frac{1}{h_{21} h_{23} (h_{11} + h_{13}) (h_{21} + h_{23})} \left[h_{23}^2 (f_{i+1,j+1,k} - f_{i-1,j+1,k}) - (h_{23}^2 - h_{21}^2) (f_{i+1,j,k} - f_{i-1,j,k}) - h_{21}^2 (f_{i+1,j-1,k} - f_{i-1,j-1,k}) \right] \quad (61)$$

$$b) \left. \frac{\partial^2 f}{\partial x_1 \partial x_2} \right|_{i,j,k} = \frac{1}{h_{11} h_{13} (h_{11} + h_{13}) (h_{21} + h_{23})} \left[h_{13}^2 (f_{i+1,j+1,k} - f_{i+1,j-1,k}) - (h_{13}^2 - h_{11}^2) (f_{i,j+1,k} - f_{i,j-1,k}) - h_{11}^2 (f_{i-1,j+1,k} - f_{i-1,j-1,k}) \right] \quad (62)$$

Fig. 2 Irregular Mesh Formulas for Mixed Central, Forward, Backward and Corner Derivatives



Second Mixed Irregular Forward Derivative With Respect to α_1 and α_2

$$c) \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \Big|_{i,j,k} = \frac{1}{h_{21} h_{23} (h_{11} - h_{12}) (h_{21} + h_{22})} \left[h_{23}^2 (f_{i+1,j+1,k} - f_{i+1,j,k} - f_{i+2,j+1,k} + f_{i+2,j,k}) - h_{21}^2 (f_{i+1,j-1,k} - f_{i+1,j,k} - f_{i+2,j-1,k} + f_{i+2,j,k}) \right] \quad (63)$$

$$d) \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \Big|_{i,j,k} = \frac{1}{h_{11} h_{13} (h_{21} - h_{22}) (h_{11} + h_{13})} \left[h_{13}^2 (f_{i+1,j+1,k} - f_{i,j+1,k} - f_{i+1,j+2,k} + f_{i,j+2,k}) - h_{11}^2 (f_{i-1,j+1,k} - f_{i,j+1,k} - f_{i-1,j+2,k} + f_{i,j+2,k}) \right] \quad (64)$$

Second Mixed Irregular Backward Derivative With Respect to α_1 and α_2

$$e) \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \Big|_{i,j,k} = \frac{-1}{h_{21} h_{23} (h_{13} - h_{14}) (h_{21} + h_{23})} \left[h_{23}^2 (f_{i-1,j+1,k} - f_{i-2,j+1,k} + f_{i-2,j,k} - f_{i-1,j,k}) - h_{21}^2 (f_{i-1,j-1,k} - f_{i-1,j,k} - f_{i-2,j-1,k} + f_{i-2,j,k}) \right] \quad (65)$$

$$f) \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \Big|_{i,j,k} = \frac{-1}{h_{11} h_{13} (h_{20} - h_{24}) (h_{11} + h_{13})} \left[h_{13}^2 (f_{i+1,j-1,k} - f_{i+1,j-2,k} + f_{i,j-2,k} - f_{i,j-1,k}) - h_{11}^2 (f_{i-1,j-1,k} - f_{i,j-1,k} - f_{i-1,j-2,k} + f_{i,j-2,k}) \right] \quad (66)$$

Second Mixed Irregular Corner Derivative With Respect To x_1 and x_2

$$g) \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{1}{h_{11} h_{12} h_{21} h_{22} (h_{22} - h_{21})} [h_{12} h_{22}^2 (f_{i+1,j+1,k} - f_{i-1,j,k} - f_{i,j+1,k} + f_{i,j,k}) - h_{11} h_{21}^2 (f_{i+2,j+2,k} - f_{i+2,j,k} - f_{i,j+2,k} + f_{i,j,k})] \quad (67)$$

Second Mixed Regular Derivatives

All of the above results can be reduced to regular derivatives with respect to either x_1 , x_2 or both coordinates by making the substitutions

$$h_{11} = h_{13} = \frac{h_{12}}{2} = \frac{h_{14}}{2} = h_1 \quad (68)$$

$$h_{21} = h_{23} = \frac{h_{22}}{2} = \frac{h_{24}}{2} = h_2 \quad (69)$$

$$h_1 = h_2 = h \quad (70)$$

The various derivatives are summarized below for the case in which all grid spacings are equal (i.e., $h_1 = h_2$).

Second Mixed Regular Central Derivative With Respect To x_1 and x_2

a), b)

$$\left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)_{i,j,k} = \frac{1}{4h^2} (f_{i+1,j+1,k} - f_{i+1,j-1,k} - f_{i-1,j+1,k} + f_{i-1,j-1,k}) \quad (71)$$

Second Mixed Regular Forward Derivative With Respect To x_1 and x_2

$$c) \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)_{i,j,k} = \frac{-1}{2h^2} (f_{i+1,j+1,k} - f_{i+2,j+1,k} - f_{i+1,j-1,k} + f_{i+2,j-1,k}) \quad (72)$$

$$d) \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)_{i,j,k} = \frac{-1}{2h^2} (f_{i+1,j+1,k} - f_{i+1,j+2,k} - f_{i-1,j+1,k} + f_{i-1,j+2,k}) \quad (73)$$

Second Mixed Regular Backward Derivative With Respect to x_1 and x_2

$$e) \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)_{i,j,k} = \frac{1}{2h^2} (f_{i-1,j+1,k} - f_{i-2,j+1,k} - f_{i-1,j-1,k} + f_{i-2,j-1,k}) \quad (74)$$

$$f) \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)_{i,j,k} = \frac{1}{2h^2} (f_{i+1,j-1,k} - f_{i+1,j-2,k} - f_{i-1,j-1,k} + f_{i-1,j-2,k}) \quad (75)$$

Second Mixed Regular Corner Derivative With Respect to x_1 and x_2

$$g) \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)_{i,j,k} = \frac{1}{4h^2} \left[8(f_{i+1,j+1,k} - f_{i+1,j,k} - f_{i,j+1,k} + f_{i,j,k}) \right. \\ \left. - (f_{i+2,j+2,k} - f_{i+2,j,k} - f_{i,j+2,k} + f_{i,j,k}) \right] \quad (76)$$

TABLE 3 COEFFICIENTS OF STRESS EQUATIONS
SPHERICAL COORDINATES

l	1	2	3	4	5	6
α_1	$\lambda + 2\mu$ \odot	λ \odot	λ \odot	0 \odot	0 \odot	0 \odot
β_1	0 \odot	0 \odot	0 \odot	μ/R \odot	0 \odot	0 \odot
γ_1	0 \odot	0 \odot	0 \odot	0 \odot	0 \odot	$\mu/(R \sin \varphi)$ \odot
δ_1	$2\lambda/R$ \odot	$2(\lambda + \mu)/R$ \odot	$2(\lambda + \mu)/R$ \odot	0 \odot	0 \odot	0 \odot
$\bar{\alpha}_1$	0 \odot	0 \odot	0 \odot	μ \odot	0 \odot	0 \odot
$\bar{\beta}_1$	λ/R \odot	$(\lambda + 2\mu)/R$ \odot	λ/R \odot	0 \odot	0 \odot	0 \odot
$\bar{\gamma}_1$	0 \odot	0 \odot	0 \odot	0 \odot	$\mu/(R \sin \varphi)$ \odot	0 \odot
$\bar{\delta}_1$	$\frac{\lambda \cot \varphi}{R}$ \odot	$\lambda \cot \varphi / R$ \odot	$(\lambda + 2\mu) \cot \varphi / R$ \odot	$-\mu/R$ \odot	0 \odot	0 \odot
$\bar{\alpha}_2$	0 \odot	0 \odot	0 \odot	0 \odot	0 \odot	μ \odot
$\bar{\beta}_2$	0 \odot	0 \odot	0 \odot	0 \odot	μ/R \odot	0 \odot
$\bar{\gamma}_2$	$\lambda/(R \sin \varphi)$ \odot	$\lambda/(R \sin \varphi)$ \odot	$(\lambda + 2\mu)/(R \sin \varphi)$ \odot	0 \odot	0 \odot	0 \odot
$\bar{\delta}_2$	0 \odot	0 \odot	0 \odot	0 \odot	$-\mu \cot \varphi / R$ \odot	$-\mu/R$ \odot

TOROIDAL COORDINATES

α_1	$\lambda + 2\mu$ \odot	λ \odot	λ \odot	0 \odot	0 \odot	0 \odot
β_1	0 \odot	0 \odot	0 \odot	μ/r \odot	0 \odot	0 \odot
γ_1	0 \odot	0 \odot	0 \odot	0 \odot	0 \odot	$\mu/(a + r \sin \varphi)$ \odot
δ_1	$\frac{\lambda(a + 2r \sin \varphi)}{r(a + r \sin \varphi)}$ \odot	$\frac{\lambda + 2\mu}{r} + \frac{\lambda \sin \varphi}{a + r \sin \varphi}$ \odot	$\frac{\lambda}{r} + \frac{(\lambda + 2\mu) \sin \varphi}{a + r \sin \varphi}$ \odot	0 \odot	0 \odot	0 \odot
$\bar{\alpha}_1$	0 \odot	0 \odot	0 \odot	μ \odot	0 \odot	0 \odot
$\bar{\beta}_1$	λ/R \odot	$(\lambda + 2\mu)/r$ \odot	λ/r \odot	0 \odot	0 \odot	0 \odot
$\bar{\gamma}_1$	0 \odot	0 \odot	0 \odot	0 \odot	$\mu/(a + r \sin \varphi)$ \odot	0 \odot
$\bar{\delta}_1$	$\frac{\lambda \cos \varphi}{a + r \sin \varphi}$ \odot	$\frac{\lambda \cos \varphi}{a + r \sin \varphi}$ \odot	$\frac{(\lambda + 2\mu) \cos \varphi}{a + r \sin \varphi}$ \odot	$-\mu/r$ \odot	0 \odot	0 \odot
$\bar{\alpha}_2$	0 \odot	0 \odot	0 \odot	0 \odot	0 \odot	μ \odot
$\bar{\beta}_2$	0 \odot	0 \odot	0 \odot	0 \odot	μ/r \odot	0 \odot
$\bar{\gamma}_2$	$\lambda/(a + r \sin \varphi)$ \odot	$\frac{\lambda}{a + r \sin \varphi}$ \odot	$\frac{\lambda + 2\mu}{a + r \sin \varphi}$ \odot	0 \odot	0 \odot	0 \odot
$\bar{\delta}_2$	0 \odot	0 \odot	0 \odot	0 \odot	$-\frac{\mu \cos \varphi}{a + r \sin \varphi}$ \odot	$-\frac{\mu \sin \varphi}{a + r \sin \varphi}$ \odot